

# Topological Symmetries of $\mathbb{R}^3$ , II

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This paper should be viewed as an extension of [KS]. It deals with orientation reversing topological actions of finite groups  $G$  on the Euclidean space  $\mathbb{R}^3$ . All actions in this paper are assumed to be effective.

**Remark 1** This paper combined with [KS] gives complete classification of finite topological symmetries of  $\mathbb{R}^3$ , namely:

**Theorem 1** *Let  $G$  be a finite group acting topologically on  $\mathbb{R}^3$ , then  $G$  is isomorphic to a finite subgroup of  $O(3)$ .*

The structure of this paper is very similar to that of [KS]. We start with some preliminary result and then define six types of Obstruction Kernels, from Type A to Type F.

Given the classification of finite groups acting (topologically) orientation preservingly on  $\mathbb{R}^3$ , it is clear the group  $G$  is an extension of such group by  $\mathbb{Z}_2$ . The Obstruction Kernels will allow us to exclude all possibilities where the group  $G$  is not contained in  $O(3)$ .

## 1 Preliminaries

**Lemma 1** *If  $f$  is an orientation reversing involution of  $S^3$  with fixed point set  $S^2$ , then  $f$  permutes the two components of  $S^3 - S^2$ .*

**Proof** Suppose the contrary holds. Let  $S^3 - S^2$  be  $A \cup B$  where  $A, B$  are the two components.  $f$  restricts to a homeomorphism on  $A$ , i.e.,  $\mathbb{Z}_2$  acts on  $A$ . Since  $A$  is acyclic,  $A^{\mathbb{Z}_2}$  is nonempty according to Smith Theory (cf. [B] p.145). This is impossible since  $(S^3)^{\mathbb{Z}_2} = S^2$ .  $\square$

**Theorem 2** *If  $D_{2n}(n > 2)$  acts on  $\mathbb{R}^3$  such that  $\mathbb{Z}_n \subseteq D_{2n}$  is the collection of orientation preserving homeomorphisms, then  $(S^3)^{D_{2n}} \cong S^1$ .*

**Proof** Assume this is not the case. Then  $(S^3)^{D_{2n}} = [(S^3)^{\mathbb{Z}_n}]^{\mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2} = S^0$ . Let  $a, b$  be standard generators of  $D_{2n}$ . For  $0 \leq i \leq n-1$ ,  $a^i b$  is an orientation reversing homeomorphism, so  $(S^3)^{\langle a^i b \rangle} \cong S^0$  or  $S^2$ .

Case 1: If for all  $i$ ,  $(S^3)^{\langle a^i b \rangle} = S^0$ , then the fixed point set of any nontrivial subgroup of  $D_{2n}$  is in  $S^1 = (S^3)^{\mathbb{Z}_n}$ . Thus  $D_{2n}$  acts on  $S^3 - S^1$  freely. But this implies (since  $S^3 - S^1$  is a homological 1-sphere) that  $D_{2n}$  is cyclic, which is impossible.

Case 2: If there exist  $0 \leq i \leq n-1$  such that  $(S^3)^{\langle a^i b \rangle} \cong S^2$ . Fix  $i$  and denote  $(S^3)^{\langle a^i b \rangle}$  as  $S_0^2$ . Now  $a(S^3)^{\langle a^i b \rangle} = (S^3)^{a\langle a^i b \rangle a^{-1}} = (S^3)^{\langle a^{i+2} b \rangle}$ , denote this set as  $S_1^2$  (as the name suggests, it is homeomorphic to  $S^2$ ).  $S_1^2 \cap S_0^2 = (S^3)^{\langle a^i b, a^{i+2} b \rangle} = (S^3)^{\langle a^2, a^i b \rangle} = [(S^3)^{\langle a^2 \rangle}]^{\mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2} = S^0$ . Note that this  $S^0$  is the fixed point set of  $D_{2n}$  and  $S^0 \subseteq S_j^2$ ,  $j = 0, 1$ .

Since  $\langle a \rangle$  is a cyclic group acting orientation preservingly,  $(S^3)^{\langle a \rangle} \cong S^1$ .  $S^0 \subseteq S^1$ , and  $S^1 - S^0$  has two components. Denote them as  $\mathbb{R}_+$  and  $\mathbb{R}_-$ .

Claim: For  $j = 0, 1$ ,  $\mathbb{R}_+$  and  $\mathbb{R}_-$  lie in different components of  $S^3 - S_j^2$ .

Proof: Suppose otherwise. The homeomorphism  $a^{i+2j}b$  fixes  $S_j^2$ , and by the preceding lemma it permutes the two components. On the other hand,  $D_{2n}/\langle a \rangle$  acts on  $(S^3)^{\langle a \rangle} = S^1$ . In particular  $a^{i+2j}b(\mathbb{R}_+ \cup \mathbb{R}_-) = \mathbb{R}_+ \cup \mathbb{R}_-$ , but  $\mathbb{R}_+ \cup \mathbb{R}_-$  is mapped to the other component of  $S^3 - S_j^2$  which contains none of  $\mathbb{R}_+$  or  $\mathbb{R}_-$ . This is a contradiction.

For  $j = 0, 1$ , let  $A_j, B_j$  be the components of  $S^3 - S_j^2$  where  $\mathbb{R}_- \subseteq A_j, \mathbb{R}_+ \subseteq B_j$ . The intersection  $S^1 \cap S_j^2 = (S^3)^{\langle a, a^{i+2j} b \rangle} = (S^3)^{D_{2n}} = S^0$ , whence  $S_1^2 - S^0 \subseteq S^3 - S_0^2$ . Now we must have either  $S_1^2 - S^0 \subseteq A_0$  or  $S_1^2 - S^0 \subseteq B_0$ .

If  $S_1^2 - S^0 \subseteq B_0$ , then  $A_0 \cup S_0^2 \subseteq A_1 \cup B_1 \cup S^0$  by taking complement. So  $A_0 \subseteq A_1 \cup B_1$ . Since  $A_0$  is connected, either  $A_0 \subseteq A_1$  or  $A_0 \subseteq B_1$ . The latter is not possible because  $\mathbb{R}_- \not\subseteq B_1$ . Thus  $A_0 \subseteq A_1$ . Obviously  $A_0 \neq A_1$  and  $A_0 \subset A_1$ . By definition,  $aA_0 = A_1$  and  $a^n A_0 = A_0$ . Then  $a^n A_0 = a^{n-1} A_1 \supset a^{n-1} A_0 \supset a^{n-2} A_0 \supset \dots \supset A_0$ , a contradiction.

This forces  $S_1^2 - S^0 \subseteq A_0$  to be the case. But this is impossible by an analogous argument as the one above.

Therefore all possibilities lead to contradictions and the initial assumption fails.  $\square$

## 2 Obstruction Kernel

**Proposition 1** (*Obstruction Kernel of Type A*) Let  $G = (\mathbb{Z}_p \oplus \mathbb{Z}_q) \rtimes_{\varphi} \mathbb{Z}_2$ , where  $\varphi(1)$  is multiplication by 1 (resp.  $-1$ ) on  $\mathbb{Z}_p$  (resp.  $\mathbb{Z}_q$ ). Then  $G$  cannot act on  $\mathbb{R}^3$ .

**Proof** Assume such action exists. Note that the subgroup  $\mathbb{Z}_q \rtimes_{\varphi} \mathbb{Z}_2$  is a dihedral group  $D_{2q}$ . If  $G$  acts orientation preservingly, this is Obstruction Kernel of Type 0 in [KS], thus impossible. So we may assume the action to be not orientation preserving.

The group  $\mathbb{Z}_p \oplus \mathbb{Z}_q$  is the only subgroup of index 2, whence it is the collection of orientation preserving homeomorphisms. In particular  $\mathbb{Z}_q$  acts orientation preservingly.

Now  $\mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_2 = \mathbb{Z}_p \oplus \mathbb{Z}_2$ . Therefore  $(S^3)^{\mathbb{Z}_p \oplus \mathbb{Z}_2} = [(S^3)^{\mathbb{Z}_2}]^{\mathbb{Z}_p}$ . Since the generator of  $\mathbb{Z}_2$  reverses orientation,  $(S^3)^{\mathbb{Z}_2} = S^2$  or  $S^0$ . In either case  $[(S^3)^{\mathbb{Z}_2}]^{\mathbb{Z}_p} = S^0$ . Now  $(S^3)^{\mathbb{Z}_p \oplus \mathbb{Z}_2} = [(S^3)^{\mathbb{Z}_p}]^{\mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2}$ . Note that  $(S^3)^{\mathbb{Z}_p} = (S^3)^{\mathbb{Z}_p \oplus \mathbb{Z}_q} = (S^3)^{\mathbb{Z}_q} \cong S^1$ , whence  $(S^1)^{\mathbb{Z}_2} = S^0 = [(S^3)^{\mathbb{Z}_q}]^{\mathbb{Z}_2} = (S^3)^{D_{2q}}$ . But by Theorem 4,  $(S^3)^{D_{2q}} = S^1$ , a contradiction.  $\square$

**Proposition 2** (*Obstruction Kernel of Type B*) Let  $G = (\mathbb{Z}_4 \oplus \mathbb{Z}_q) \rtimes_{\varphi} \mathbb{Z}_2$ ,  $q$  odd prime,  $\varphi(1)$  is multiplication by 1 (resp.  $-1$ ) on  $\mathbb{Z}_4$  (resp.  $\mathbb{Z}_q$ ). Then there is no action of  $G$  on  $\mathbb{R}^3$  such that  $\mathbb{Z}_4 \oplus \mathbb{Z}_q$  is the collection of orientation preserving homeomorphisms.

**Proof** Assume such action exists. By Theorem 2, to produce a contradiction it suffices to prove that  $(S^3)^{\mathbb{Z}_q \rtimes_{\varphi} \mathbb{Z}_2} = S^0$  ( $\mathbb{Z}_q \rtimes_{\varphi} \mathbb{Z}_2$  is dihedral and  $\mathbb{Z}_q$  is the orientation preserving subgroup). We start with computations of the fixed point sets of various subgroups of  $G$ .

Since  $\mathbb{Z}_4 \oplus \mathbb{Z}_q$  is cyclic, we have  $(S^3)^{\mathbb{Z}_4 \oplus \mathbb{Z}_q} = (S^3)^{\mathbb{Z}_q} \cong S^1$ .

The fixed point set  $(S^3)^{\mathbb{Z}_q \rtimes_{\varphi} \mathbb{Z}_2} = [(S^3)^{\mathbb{Z}_q}]^{\mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2}$  is either  $S^1$  or  $S^0$ .

The standard copy of  $\mathbb{Z}_2$  acts orientation reversingly, whence  $(S^3)^{\mathbb{Z}_2} = S^2$  or  $S^0$ .

Case 1: If  $(S^3)^{\mathbb{Z}_2} = S^0$ , then this forces  $(S^3)^{\mathbb{Z}_q \rtimes_{\varphi} \mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2}$  to be  $S^0$  and we obtain the contradiction we are looking for.

Case 2: If  $(S^3)^{\mathbb{Z}_2} = S^2$ , then  $(S^3)^{\mathbb{Z}_4 \rtimes_{\varphi} \mathbb{Z}_2} = (S^3)^{\mathbb{Z}_4 \oplus \mathbb{Z}_2} = [(S^3)^{\mathbb{Z}_2}]^{\mathbb{Z}_4} = (S^2)^{\mathbb{Z}_4}$ . According to [E], any action on  $S^2$  is conjugate to an orthogonal one. Thus  $(S^2)^{\mathbb{Z}_4}$  is either  $S^0$  or empty. On the other hand, we have computed that  $(S^3)^{\mathbb{Z}_q \rtimes_{\varphi} \mathbb{Z}_2}$  is either  $S^1$  or  $S^0$ . Combining the two results,  $(S^3)^{\mathbb{Z}_q \rtimes_{\varphi} \mathbb{Z}_2} = S^0$ .  $\square$

**Proposition 3** (*Obstruction Kernel of Type C*) Let  $G = \mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_{2^{k+1}}$ ,  $k \geq 1$ ,  $p$  odd prime,  $\varphi(1)$  is multiplication by  $-1$ . Then  $G$  cannot act on  $\mathbb{R}^3$ .

**Proof** Assume such action exists. If the action is orientation preserving, then  $G$  is Obstruction Kernel of Type 2 as in [KS], which is impossible. So it to consider the case where the action is not orientation preserving.

It is not hard to see that  $\mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_{2^k} \cong \mathbb{Z}_{2^k p}$  is the only subgroup of  $G$  with index 2. Thus this subgroup has to be the collection of homeomorphisms preserving orientation.

Let  $b = (0, 1) \in \mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_{2^{k+1}}$ . Then  $b$  reverses orientation of  $\mathbb{R}^3$ , whence  $(S^3)^{\langle b \rangle} \cong S^0$  or  $(S^3)^{\langle b \rangle} \cong S^2$ .

Case 1:  $(S^3)^{\langle b \rangle} \cong S^2$ . In this case  $(S^3)^{\langle b^2 \rangle} \supseteq (S^3)^{\langle b \rangle} = S^2$ . But  $b^2$  is orientation preserving and thus  $(S^3)^{\langle b^2 \rangle} \cong S^1$ , a contradiction. So this case is not possible.

Case 2:  $(S^3)^{\langle b \rangle} \cong S^0$ . In this case  $(S^3)^G = S^0$ . Since  $\mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_{2^k} \cong \mathbb{Z}_{2^k p}$  acts orientation preservingly,  $(S^3)^{\mathbb{Z}_{2^k p}} \cong S^1$ . The quotient  $G/\mathbb{Z}_{2^k p} \cong \mathbb{Z}_2$  acts on this copy of  $S^1$ , so  $G = \mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_{2^{k+1}}$  acts on the complement  $S^3 - S^1$ . The fixed point set of any nontrivial subgroup  $H$  of  $G$  is either  $S^1$  (when  $\subseteq \mathbb{Z}_{2^k p}$ ) or

$S^0$  (when  $H \not\subseteq \mathbb{Z}_{o^k p}$ ). Therefore the restriction to  $S^3 - S^1$  is free. As before, this implies that  $G$  is cyclic, which is obviously not the case.

Thus either case leads to a contradiction and the proposition is proven.  $\square$

**Proposition 4** (*Obstruction Kernel of Type D*) Let  $G = \mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_2$ ,  $p$  odd prime,  $p \equiv 1 \pmod{4}$ ,  $\varphi(1) = n \in \mathbb{Z}_p^*$  where  $n^2 = -1 \pmod{p}$ . Then  $G$  cannot act on  $\mathbb{R}^3$ .

**Proof** The proof of Obstruction Kernel of Type 5 in [KS] carries verbatim.  $\square$

**Proposition 5** (*Obstruction Kernel of Type E*) Let  $G = (\mathbb{Z}_p \oplus \mathbb{Z}_4) \rtimes_{\varphi} \mathbb{Z}_2$ ,  $p$  odd prime,  $\varphi(1)$  is multiplication by 1 (resp.  $-1$ ) on  $\mathbb{Z}_p$  (resp.  $\mathbb{Z}_4$ ), then  $G$  cannot act on  $\mathbb{R}^3$  such that the collection of orientation preserving homeomorphisms is  $\mathbb{Z}_p \oplus \mathbb{Z}_4$ .

**Proof** Assume such action exists. Note that  $\mathbb{Z}_4 \rtimes_{\varphi} \mathbb{Z}_2 \cong D_8$ . To produce a contradiction, it suffices (by Theorem 2) to prove  $(S^3)^{D_8} \cong S^0$ .

By our assumption that the action restricted to  $\mathbb{Z}_p \oplus \mathbb{Z}_4 \cong \mathbb{Z}_{4p}$  is orientation preserving, we have  $(S^3)^{\mathbb{Z}_p} = (S^3)^{\mathbb{Z}_{4p}} = (S^3)^{\mathbb{Z}_4} \cong S^1$ .

The generator of  $\mathbb{Z}_2$  reverses orientation, whence  $(S^3)^{\mathbb{Z}_2} \cong S^2$  or  $S^0$ .

If  $(S^3)^{\mathbb{Z}_2} = S^0$ , then  $(S^3)^{D_8} = [(S^3)^{\mathbb{Z}_4}]^{\mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2} = S^0$ , and we are done.

If  $(S^3)^{\mathbb{Z}_2} = S^2$ , consider the subgroup  $\mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_2 \subseteq G$ . By definition it is isomorphic to  $\mathbb{Z}_p \oplus \mathbb{Z}_2$ . And  $(S^3)^{\mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_2} = (S^3)^{\mathbb{Z}_p \oplus \mathbb{Z}_2} = [(S^3)^{\mathbb{Z}_2}]^{\mathbb{Z}_p} = (S^2)^{\mathbb{Z}_p} = S^0$  since any action on  $S^2$  is conjugate to an orthogonal one. On the other hand,  $(S^3)^{\mathbb{Z}_p \rtimes_{\varphi} \mathbb{Z}_2} = [(S^3)^{\mathbb{Z}_p}]^{\mathbb{Z}_2} = (S^1)^{\mathbb{Z}_2}$ . Thus  $(S^1)^{\mathbb{Z}_2} = S^0$ . Since  $(S^1)^{\mathbb{Z}_2}$  is also  $[(S^3)^{\mathbb{Z}_4}]^{\mathbb{Z}_2} = (S^3)^{D_8}$ , we obtain  $(S^3)^{D_8} = S^0$ .  $\square$

**Proposition 6** (*Obstruction Kernel of Type F*)  $G = Q_{4m}$  (the generalized quaternion group) cannot act on  $\mathbb{R}^3$ .

**Proof** The proof follows verbatim from the proof of Obstruction Kernel of Type 3 (c.f. [KS]), using the remark following Theorem 2 in the same paper and Obstruction Kernel of Type C.  $\square$

### 3 The Cyclic Case

In [KS], we considered extensions of (finite) subgroups of  $\text{SO}(3)$  by  $\mathbb{Z}_p$ ,  $p$  prime. In the following sections, the algebraic aspects of the situations are almost the same as their counterparts in the previous paper. That paper has given an algebraic description to the possible results of extensions. Thus we will not repeat the algebra part of those proofs, but to filter them with the new obstruction kernels.

**Theorem 3** *If  $G$  acts on  $\mathbb{R}^3$  such that orientation preserving subgroup is cyclic, then  $G$  is isomorphic to a subgroup of  $O(3)$ .*

**Proof** It suffice to consider the case where the orientation preserving subgroup is of index 2.

There is an short exact sequence

$$0 \longrightarrow \mathbb{Z}_n \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

where  $\mathbb{Z}_n$  is the subgroup of orientation preserving homeomorphisms in  $G$ .

The algebraic possibilities are known (c.f. [KS] Proposition 3,4,5). We will investigate then in a way analogous to [KS]. Let  $n = 2^k m$ ,  $m$  odd. Let  $\varphi$  be the induced action of  $\mathbb{Z}_2$  on  $\mathbb{Z}_n$ . Let  $m = P \cdot Q$  where  $\varphi(1)$  restrict to a multiplication by  $+1$ (resp.  $-1$ ) on  $\mathbb{Z}_P$ (resp.  $\mathbb{Z}_Q$ ) (c.f. [KS])

Case 1:  $k = 0$  ( $n$  is odd).

As in Proposition 3 of [KS],  $G$  is either cyclic, dihedral or contains an Obstruction Kernel of Type A. In the last case  $G$  cannot act. So  $G \subseteq O(3)$ .

Case 2:  $k = 1$  ( $n = 2m$ ,  $m$  odd)

We proceed as in Proposition 4 in [KS].

i) Split Case:

If  $Q = 1$ ,  $G = \mathbb{Z}_n \oplus \mathbb{Z}_2$ , which is a subgroup of  $O(3)$ .

If  $P = 1$ ,  $G$  is dihedral, thus  $G \subseteq O(3)$

If neither  $P$  nor  $Q$  is 1,  $G$  contains an Obstruction Kernel of Type A, which is impossible.

ii) Non-split Case:

If  $Q = 1$ , then  $G$  is cyclic and  $G \subseteq O(3)$ .

If  $Q > 1$ , then  $G$  contains an Obstruction Kernel of Type C, a contradiction.

In sum, for  $n = 2m$ ,  $m$  odd,  $G \subseteq O(3)$ .

Case 3:  $k \geq 2$

We argue as in Proposition 5 of [KS]. There are four possibilities for  $\varphi$  restricted on the standard copy of  $\mathbb{Z}_{2^k}$  in  $\mathbb{Z}_n$ .

i)  $\varphi(1)$  is multiplication by 1.

Split Case:  $G = \mathbb{Z}_{2^k m} \rtimes_{\varphi} \mathbb{Z}_2$ . Either  $P$  or  $Q$  since otherwise there will be an Obstruction Kernel of Type A in  $G$ .

If  $Q = 1$ , then  $G = \mathbb{Z}_n \oplus \mathbb{Z}_2$ , whence  $G \subseteq O(3)$ .

If  $P = 1$ , then  $G = (\mathbb{Z}_{2^k} \oplus \mathbb{Z}_Q) \rtimes_{\varphi} \mathbb{Z}_2$ , thus contains Obstruction Kernel of Type B. This is impossible.

Non-split Case: In this case  $G \cong (\mathbb{Z}_P \oplus \mathbb{Z}_Q) \rtimes_{\phi} \mathbb{Z}_{2^{k+1}}$  where  $\phi(1)$  is multiplication by 1 (resp.  $-1$ ) on  $\mathbb{Z}_P$  (resp.  $\mathbb{Z}_Q$ )

If  $Q = 1$ ,  $G$  is cyclic thus isomorphic to a subgroup of  $O(3)$ .

If  $Q > 1$ ,  $G$  contains an Obstruction Kernel of Type C.

ii)  $\varphi(1)$  is multiplication by  $-1$

Split Case:  $G \cong \mathbb{Z}_n \rtimes_{\varphi} \mathbb{Z}_2$ . Again either  $P = 1$  or  $Q = 1$  since an Obstruction Kernel of Type A will show up otherwise.

If  $Q = 1$ , then  $G$  contains an Obstruction Kernel of Type E, therefore this case is excluded.

If  $P = 1$ ,  $G$  is dihedral.

Non-Split Case: In such case  $G \cong (\mathbb{Z}_P \oplus \mathbb{Z}_Q) \rtimes_{\phi} Q_{4m}$  where for some  $\phi$ , but  $Q_{4m}$  is Obstruction Kernel of Type F, which implies this case cannot occur.

iii)  $\varphi(1)$  is multiplication by  $2^{k-1} + 1$ , then  $G$  contains  $\mathbb{Z}_{2^k} \rtimes_{\varphi} \mathbb{Z}_2$ , a 2-group. This however contradicts the remark following Theorem 2 of [KS]. Thus this case is impossible.

iv)  $\varphi(1)$  is multiplication by  $2^{k-1} - 1$ . A same argument as above produces a contradiction.

In all the possible cases above,  $G$  has to be isomorphic to a subgroup of  $O(3)$ .  $\square$

**Remark 2** The proof actually shows that  $G$  is either  $\mathbb{Z}_{2n}$ ,  $\mathbb{Z}_n \oplus \mathbb{Z}_2$  or  $D_{2n}$ .

## 4 The Dihedral Case

The proof of the dihedral case follows the spirit of Proposition 8 and 9 in [KS].

**Theorem 4** *If  $G$  acts on  $\mathbb{R}^3$  such that the subgroup of orientation preserving homeomorphisms is  $D_{2n}$ ,  $n$  odd,  $n \geq 3$ , then  $G$  is isomorphic to a subgroup of  $O(3)$ .*

**Proof** It suffice to consider the case where the action is not orientation preserving. The first half of the proof of Proposition 8 of [KS] carries verbatim. We have two cases (notations are borrowed from that proof):

Case 1: If  $(2a_1, 2a_2, \dots, 2a_n) = (0, 0, \dots, 0)$

In this case the short exact sequence

$$0 \longrightarrow D_{2n} \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

splits and thus  $G \cong D_{2n} \rtimes_{\varphi} \mathbb{Z}_2$ ,  $\varphi$  as defined in Proposition 8 of [KS].

We have  $b_i = \pm 1$  for all  $i$ . There are three subcases:

i) Both  $\pm 1$  appears. In this case  $G$  contains an Obstruction Kernel of Type A, which is impossible.

ii)  $b_i = 1$  for all  $i$ . then the action  $\varphi$  is trivial on the cyclic subgroup  $\mathbb{Z}_n$ .  $\varphi$  is always trivial on the period 2 generator  $b$  of  $D_{2n}$  by definition. Thus  $\varphi$  is trivial and  $G \cong D_{2n} \times \mathbb{Z}_2$ , a subgroup of  $O(3)$ .

iii)  $b_i = -1$  for all  $i$ , in this case  $G$  has been computed to be dihedral.

Case 2: if  $(2a_1, 2a_2, \dots, 2a_n) = \left(\frac{p_1-1}{2}p_1^{n_1-1}, \dots, \frac{p_k-1}{2}p_k^{n_k-1}\right)$

In this case  $G$  (as computed in [KS]) contains an Obstruction Kernel of Type D, which is a contradiction.

Summing the above results, we see in all possible cases  $G \subseteq O(3)$ .  $\square$

**Proposition 7** *Suppose  $G$  acts on  $\mathbb{R}^3$  such that the subgroup of orientation preserving homeomorphisms is  $D_{2n}$ ,  $n$  even. If  $n > 2$ , then the extension  $0 \rightarrow D_{2n} \rightarrow G \rightarrow \mathbb{Z}_2 \rightarrow 0$  has to split.*

**Proof** Assume this is not the case.

Take the Sylow 2-subgroup of  $D_{2n}$ . It must be a copy of  $D_{2^{l+1}}$ . Let  $P$  be the Sylow 2-subgroup of  $G$  containing this  $D_{2^{l+1}}$ . As a two group,  $P \subseteq O(3)$ , whence it is either  $\mathbb{Z}_{2^{l+2}}$ ,  $\mathbb{Z}_{2^{l+1}} \oplus \mathbb{Z}_2$ ,  $D_{2^{l+2}}$  or  $D_{2^{l+1}} \times \mathbb{Z}_2$ . Containing  $D_{2^{l+1}}$ ,  $P$  is not cyclic. It cannot be  $D_{2^{l+2}}$  or  $D_{2^{l+1}} \times \mathbb{Z}_2$  either since that would make the extension  $0 \rightarrow D_{2n} \rightarrow G \rightarrow \mathbb{Z}_2 \rightarrow 0$  splits ( $P - D_{2^{l+1}}$  contains an element of order 2). Thus  $P \cong \mathbb{Z}_{2^{l+1}} \oplus \mathbb{Z}_2$ . This group is cyclic, and the same has to be true for  $D_{2^{l+1}}$ , whence  $l = 1$ . In other word,  $n = 2m$ ,  $m$  odd.

We have seen that  $\text{Aut } D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$ . It is not hard to compute that  $\text{Inn } D_{2n} = \mathbb{Z}_m \rtimes \{\pm 1\} \subseteq \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$  where the embedding  $\mathbb{Z}_m \subseteq \mathbb{Z}_n$  is canonical. Thus  $\text{Out } D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_n^* / \mathbb{Z}_m \rtimes \{\pm 1\}$ .

The exact sequence

$$0 \longrightarrow D_{2n} \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

induces an abstract kernel  $\phi : \mathbb{Z}_2 \rightarrow \text{Out } D_{2n}$ .  $\phi(1)$  is represented by any conjugation of an element of  $G - D_{2n}$  on  $D_{2n}$ .

Consider the subgroup  $\{e, a^m, b, a^m b\} \subseteq D_{2n}$  where  $e$  stands for identity. This is a copy of  $D_4$ . Let  $P'$  be a Sylow 2-subgroup of  $G$  containing  $D_4$ .  $P' \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$ . In particular,  $P'$  is abelian. Take  $x \in P' - D_4$ . the conjugation of  $x$  on  $D_4$  is trivial. Now  $x \in G - D_{2n}$ . Let  $(t, s) \in \mathbb{Z}_n \rtimes \mathbb{Z}_n^* = \text{Aut } D_{2n}$  be the conjugation by  $x$ . This automorphism sent  $b$  to  $a^t b$ . Thus  $t = 0 \in \mathbb{Z}_n$ . So  $\phi(1)$  can be represented by  $(0, s)$ .  $\phi(1)^2 = 0$  implies  $s^2 = \pm 1 \in \mathbb{Z}_n^*$ .

Now assume  $m = p_1^{n_1} \dots p_k^{n_k}$  is the prime decomposition, then

$$\mathbb{Z}_n^* = \mathbb{Z}_2^* \times \mathbb{Z}_{p_1^{n_1}}^* \times \dots \times \mathbb{Z}_{p_k^{n_k}}^* \cong \mathbb{Z}_{p_1^{n_1}}^* \times \dots \times \mathbb{Z}_{p_k^{n_k}}^*$$

Let  $(b_1, \dots, b_k)$  be the element in the rightmost group above corresponding to  $s$ .

Since  $D_{2n}$  is centerless for  $n > 2$ , then (as computed in Proposition 8 of [KS]) each abstract kernel corresponds to one and only one extension.

Case 1: If  $s^2 = 1$ , then  $b_i = \pm 1$ . Consider the homomorphism

$$\varphi : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_n \rtimes \mathbb{Z}_n^* = \text{Aut } D_{2n}$$

where  $\varphi(1) = (0, s)$ . The extension

$$0 \longrightarrow D_{2n} \longrightarrow D_{2n} \rtimes_{\varphi} \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

induces the abstract kernel  $\phi$ . By uniqueness this split extension is equivalent to  $0 \longrightarrow D_{2n} \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 0$ , contradicting to the non-splitting assumption.

Case 2: If  $s^2 = -1$ , then  $b_i = m_i$  where  $m_i^2 \equiv -1 \pmod{p_i^{n_i}}$ . Consider

$$f : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_n^* \cong \prod_i \mathbb{Z}_{p_i^{n_i}}^*$$

where  $f(1) = \prod_i m_i$ . The canonical subgroup  $\mathbb{Z}_n \rtimes_f \mathbb{Z}_2 \subseteq \mathbb{Z}_n \rtimes_f \mathbb{Z}_4$  is dihedral since  $m^2 \equiv -1$ , and

$$0 \longrightarrow D_{2n} \longrightarrow \mathbb{Z}_n \rtimes_f \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

induces the abstract kernel  $\phi$ . By uniqueness  $G \cong \mathbb{Z}_n \rtimes_f \mathbb{Z}_4$ . This however contains an Obstruction Kernel of Type D, which is thus impossible.

In sum, either case leads to a contradiction and the assumption fails.  $\square$

**Theorem 5** *If  $G$  acts on  $\mathbb{R}^3$  such that the subgroup of orientation preserving homeomorphisms is  $D_{2n}$ ,  $n$  even, then  $G$  is isomorphic to a subgroup of  $O(3)$ .*

**Proof** If  $n = 2$ , then  $G$  is a 2-group and the result is known. So it suffice to consider the case where  $n > 2$ .

By the preceding proposition, the extension  $0 \rightarrow D_{2n} \rightarrow G \rightarrow \mathbb{Z}_2 \rightarrow 0$  splits. Thus  $G \cong D_{2n} \rtimes_{\varphi} \mathbb{Z}_2$  for some  $\varphi : \mathbb{Z}_2 \longrightarrow \text{Aut } D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_n^*$ . Let  $\varphi(1) = (t, s)$ .

Consider the subgroup  $\mathbb{Z}_n \rtimes_{\varphi} \mathbb{Z}_2 \subseteq G$ . By Remark 2, this subgroup is isomorphic to either  $\mathbb{Z}_{2n}$ ,  $\mathbb{Z}_n \oplus \mathbb{Z}_2$  or  $D_{2n}$ . Since  $n$  is even,  $\mathbb{Z}_n \rtimes_{\varphi} \mathbb{Z}_2$  cannot be cyclic. Thus there are two possibilities.

Case 1:  $\mathbb{Z}_n \rtimes_{\varphi} \mathbb{Z}_2 \cong \mathbb{Z}_n \oplus \mathbb{Z}_2$

In this case  $s = 1$ .  $\varphi(1)^2 = (t, 1)^2 = (2t, 1) = (0, 1)$ . Thus  $t = 0$  or  $t = m$ ,  $m = \frac{n}{2}$ .

i) If  $t = 0$ , then  $\varphi(1) = (0, 1)$ , which is the identity isomorphism of  $D_{2n}$ , thus  $G = D_{2n} \times \mathbb{Z}_2$ .

ii) If  $t = m$ , then  $\varphi(1) = (m, 1)$ . Consider the subgroup  $\{e, a^m, b, a^m b\}$  of  $D_{2n}$ . This is isomorphic to  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ , and  $\varphi(1)$  restrict to an isomorphism of this group. It is not hard to see the resulted  $(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes_{\varphi} \mathbb{Z}_2$  is isomorphic to  $Q_8$ . But this is Obstruction Kernel of Type F, a contradiction.

Case 2:  $\mathbb{Z}_n \rtimes_{\varphi} \mathbb{Z}_2 \cong D_{2n}$

In this case  $s = -1$ . We divide the discussion by parity of  $t$ .

i) If  $t$  is odd. Then  $G$  is dihedral as computed in Proposition 9 of [KS].

ii) If  $t$  is even. Let  $r = \frac{t}{2}$ . The element  $(a^r b, 1)$  is of order 2, and  $(a^r b, 1) \in G - D_{2n}$ . Thus  $G \cong D_{2n} \rtimes_{\phi} \mathbb{Z}_2$  where  $\phi(1)$  is the conjugation by  $(a^r b, 1)$ . An easy computation shows that

$$\begin{aligned} (a^r b, 1) \cdot (a, 0) \cdot (a^r b, 1) &= (a, 0) \\ (a^r b, 1) \cdot (0, 1) \cdot (a^r b, 1) &= (0, 1) \end{aligned}$$

Thus  $\phi(1)$  is the identity and  $G \cong D_{2n} \times \mathbb{Z}_2$ .

We have seen that in all possible cases,  $G \subseteq O(3)$ .  $\square$



**Remark 3** The last case in the proof actually gives an alternative proof to Obstruction Kernel of Type 6 in [KS], since  $G$  then contains a copy of  $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ , which is not allowed in the orientation preserving case.

Summing up the above results, we have:

**Corollary 1** *If  $G$  acts on  $\mathbb{R}^3$  such that the subgroup of orientation preserving homeomorphisms is dihedral, then  $G \subseteq O(3)$ .*

## 5 The $A_4$ Case

**Theorem 6** *If  $G$  acts on  $\mathbb{R}^3$  such that the subgroup of orientation preserving homeomorphisms is  $A_4$ , then  $G \subseteq O(3)$ .*

**Proof** There are up to isomorphism 15 groups of order 24, among which only  $A_4 \times \mathbb{Z}_2$  and  $S_4$  contains  $A_4$ . Both are subgroup of  $O(3)$ .  $\square$

## 6 The $S_4$ Case

**Theorem 7** *If  $G$  acts on  $\mathbb{R}^3$  such that the subgroup of orientation preserving homeomorphisms is  $S_4$ , then  $G \subseteq O(3)$ .*

**Proof** The only extension of  $S_4$  by  $\mathbb{Z}_2$  is  $S_4 \times \mathbb{Z}_2$  (cf. [KS]).  $\square$

## 7 The $A_5$ Case

**Theorem 8** *If  $G$  acts on  $\mathbb{R}^3$  such that the subgroup of orientation preserving homeomorphisms is  $A_5$ , then  $G \subseteq O(3)$ .*

**Proof** There are only two extensions of  $A_5$  by  $\mathbb{Z}_2$ :  $A_5 \times \mathbb{Z}_2$  and  $S_5$  (cf. [KS]). It suffice to prove that  $S_5$  cannot act on  $\mathbb{R}^3$ . Now suppose there is such an action.

There is a subgroup of  $S_5$  isomorphic to  $GA(1, 5) \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_5^*$ . This group cannot be embedded in  $A_5$ , thus the action restrict to an orientation reversing one on it (alternatively one can use the proof in [KS] to show that this group cannot act orientation preservingly). There is only one subgroup in  $\mathbb{Z}_5 \rtimes \mathbb{Z}_5^*$  on index 2 and it is a copy of  $D_{10}$ . This  $D_{10}$  then has to be the subgroup of orientation preserving homeomorphisms.

By the dihedral case,  $\mathbb{Z}_5 \rtimes \mathbb{Z}_5^*$  has to be a subgroup of  $O(3)$ . It is not abelian, thus has to be either  $D_{20}$  or  $D_{10} \times \mathbb{Z}_2$ . Neither can be the case (the former is discussed in [KS], while the latter can be done by comparing Sylow 2-subgroups). This contradicts with the assumption.  $\square$

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